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Fit to gluon propagator and Gribov formula

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Abstract

We report a numerical study of $SU(2)$ lattice gauge theory in the minimal Coulomb gauge at $\beta = 2.2$ and 9 different volumes. After extrapolation to infinite volume, our fit agrees with a lattice discretization of Gribov's formula for the transverse equal-time would-be physical gluon propagator, that *vanishes* like $|\mathbf{k}|$ at $\mathbf{k} = \mathbf{0}$, whereas the free equal-time propagator $(2|\mathbf{k}|)^{-1}$ *diverges*. Our fit lends reality to a confinement scenario in which the would-be physical gluons leave the physical spectrum while the long-range Coulomb force confines color.

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1. Introduction

In QCD a rectangular Wilson loop $W(R,T)$ of dimension $R \times T$ has, asymptotically at large T, the form $W(R,T) \sim \exp[-T V_W(R)]$, where $V_W(R)$ is the Wilson potential. If dynamical quarks are present, they are polarized from the vacuum, and $V_W(R)$ represents the interaction energy of a pair of mesons at separation R. In this case $V_W(R)$ is not a color-confining potential, but rather a QCD analog of the Van der Waals potential between neutral atoms. It clearly cannot serve as an order parameter for confinement of color in the presence of dynamical quarks, and we turn instead to gauge-dependent quantities to characterize color confinement.

A particularly simple confinement scenario [1,2] is available in the minimal Coulomb gauge.[1] It attributes confinement of color to the *enhancement* at long range of the color-Coulomb potential $V(R)$. This is the instantaneous part of the 4–4 component of the gluon propagator, $D_{44}(\mathbf{x},t) = V(|\mathbf{x}|)\delta(t) + P(\mathbf{x},t)$, where the vacuum polarization term $P(\mathbf{x},t)$ is less singular at $t = 0$. At the same time, the disappearance of gluons from the physical spectrum is manifested by the *suppression* at $\mathbf{k} = \mathbf{0}$ of the propagator $D_{ij}(\mathbf{k},k_4)$ of 3-dimensionally transverse would-be physical gluons. This behavior is clearly exhibited in Fig. 1 which displays the equal-time propagators $D^{\mathrm{tr}}(\mathbf{k})$ and $D_{44}(\mathbf{k})$.

[1] The minimal lattice Coulomb gauge is obtained by first minimizing $-\sum_{x,i} \mathrm{Tr}^g\, U_{x,i}$ with respect to all local gauge transformations $g(x)$, and then minimizing $-\sum_x \mathrm{Tr}^g\, U_{x,4}$ with respect to all $\mathbf{x}$-independent but x_4-dependent gauge transformations $g(x_4)$. This makes the 3-vector potential A_i, for $i = 1,2,3$ transverse, $\partial_i A_i = 0$, so $A_i = A_i^{\mathrm{tr}}$. Moreover, the Coulomb gauge is the finite limit of renormalizable gauges [3].

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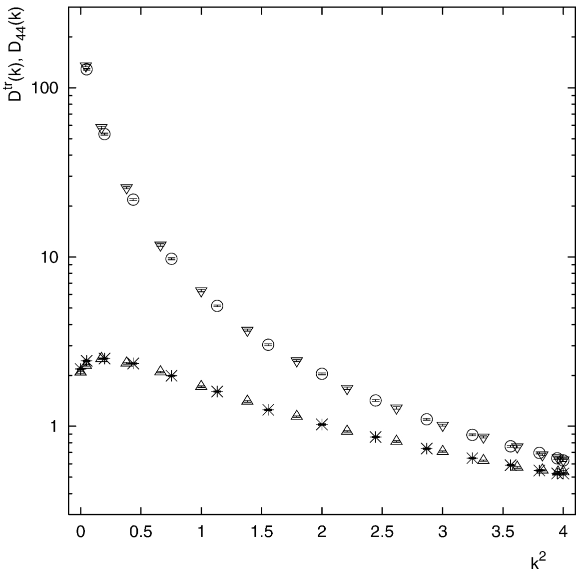

Fig. 1. Plot of the gluon propagators $D^{\mathrm{tr}}(\mathbf{k})$ and $D_{44}(\mathbf{k})$ as a function of the square of the lattice momentum $\mathbf{k}^2$ for $L = 28$ (symbols $*$ and $\bigcirc$, respectively) and $L = 30$ (symbols $\triangle$ and ∇, respectively). Notice the logarithmic scale in the y axis. Error bars are one standard deviation.

We conjecture that the color-Coulomb potential $V(R)$ is linearly rising at large R (at least when asymptotic freedom holds) and that this linear rise may serve as an order parameter for color confinement in the presence or absence of dynamical quarks [2]. As a practical matter, $V(R)$ is the starting point for calculations of bound states such as heavy quarkonium [4]. Remarkably, $V(R)$ is a renormalization-group invariant [2] in the sense that it is independent of the cut-off Λ and of the renormalization mass μ. Its Fourier transform $\widetilde{V}(\mathbf{k})$ may serve to define the running coupling constant of QCD by $\mathbf{k}^2\widetilde{V}(\mathbf{k}) = x_0 g^2_{\mathrm{coul}}(|\mathbf{k}|)$. Here $x_0 = 12N/(11N - 2N_f)$ was calculated in [5], and $g_{\mathrm{coul}}(|\mathbf{k}|)$ satisfies the perturbative renormalization group equation $|\mathbf{k}|\partial g_{\mathrm{coul}}/\partial|\mathbf{k}| = \beta_{\mathrm{coul}}(g_{\mathrm{coul}})$. It has the leading asymptotic behavior $g^2_{\mathrm{coul}}(|\mathbf{k}|) \sim [2b_0 \ln(|\mathbf{k}|/\Lambda_{\mathrm{coul}})]^{-1}$, where $b_0 = (4\pi)^{-2}(11N - 2N_f)/3$, and $\Lambda_{\mathrm{coul}} \propto \Lambda_{\mathrm{QCD}}$ is a finite QCD mass scale. These formulas allow one to determine the running coupling constant of QCD from a numerical evaluation of the equal-time 2-point function D_{44} in the minimal Coulomb gauge. By contrast, in Lorentz-covariant gauges, the coupling constant $g_r(\mu)$ requires a numerical evaluation of the 3-point vertex function [6]. Numerical studies of the gluon propagator in Landau gauge may be found in [7,8].

A less intuitive but equally striking prediction concerns the 3-dimensionally transverse, would-be physical gluon propagator $D_{ij}(\mathbf{k}, k_4) = (\delta_{ij} - \hat{k}_i\hat{k}_j) \times D^{\mathrm{tr}}(\mathbf{k}, k_4)$, whose equal-time part is given by $D^{\mathrm{tr}}(\mathbf{k}) = (2\pi)^{-1}\int dk_4\, D^{\mathrm{tr}}(\mathbf{k}, k_4)$. It was proven [9] as a consequence of the Gribov horizon that, for infinite spatial lattice volume L^3, $D^{\mathrm{tr}}(\mathbf{k})$ *vanishes* at $\mathbf{k} = \mathbf{0}$, $\lim_{|\mathbf{k}|\to 0} D^{\mathrm{tr}}(\mathbf{k}) = 0$, although the rate of approach of $D^{\mathrm{tr}}(\mathbf{k}, L)$ to 0, as a function of $\mathbf{k}$ or L, was not established, nor was it determined whether the renormalized gluon propagator also shares this property. This is in marked contrast to the free massless propagator $(\mathbf{k}^2 + k_4^2)^{-1}$ which at equal-time is given by $(2\pi)^{-1}\int dk_4\, (\mathbf{k}^2 + k_4^2)^{-1} = (2|\mathbf{k}|)^{-1}$, that *diverges* at $\mathbf{k} = \mathbf{0}$. Gribov [1] obtained, for the energy of a gluon of momentum $\mathbf{k}$, the approximate expression $E(\mathbf{k}) = |\mathbf{k}|^{-1}[(\mathbf{k}^2)^2 + M^4]^{1/2}$. This gives for the equal-time, 3-dimensionally transverse, would-be physical equal-time gluon propagator the approximate formula [2]

$$D^{\mathrm{tr}}(\mathbf{k}) = \left[2E(\mathbf{k})\right]^{-1} = \left(|\mathbf{k}|/2\right)\left[\left(\mathbf{k}^2\right)^2 + M^4\right]^{-1/2}. \quad (1)$$

In the absence of an estimate of corrections, one does not know how accurate this formula may be. Its accuracy, and the crucial question of the extrapolation to large L of $D^{\mathrm{tr}}(\mathbf{k}, L)$ will be addressed in the fits reported here.

We wish to confront Gribov's theory of confinement in the minimal Coulomb gauge [1,2] with data from our numerical study of $SU(2)$ lattice gauge theory, without quarks, in the minimal Coulomb gauge at $\beta = 2.2$. Data were taken at 9 different lattice volumes L^4, with $L = 14, 16, 18, \ldots, 30$ in order to extrapolate to infinite lattice volume. The lattice Coulomb gauge is more easily accessible to numerical study than the Landau gauge because each time-slice contributes separately to the numerical average which, for a lattice of volume 30^4, gives a factor of 30 gain. Details of the numerical simulation are described in [12]. (For this study we have, however, improved the statistics for the lattice volume 18^4.) The minimal Coulomb gauge is obtained by an over-relaxation algorithm which, in

[2] There are corresponding results for the Landau gauge [1,9,10]. Also in Landau gauge, recent solutions of the Schwinger–Dyson for the gluon propagator $D(k)$ vanish at $k = 0$ [11].

general, leads to some one of the relative minima of the minimizing functional (see footnote 1), each of which is a different Gribov copy. As discussed in the Appendix of [12], the gluon propagator in Coulomb gauge is insensitive, within statistical uncertainty, as to which Gribov copy is attained, in agreement with similar results in the Landau gauge [8,13].

We reported fits of $\widetilde{V}(\mathbf{k})$ and $D^{\mathrm{tr}}(\mathbf{k}, L)$ in [12]. Our data clearly show that $\widetilde{V}(\mathbf{k})$ is more singular than $1/\mathbf{k}^2$ at low $\mathbf{k}$, which indeed corresponds to a long-range color-Coulomb potential. (A linearly rising potential $V(R)$ corresponds to $\widetilde{V}(\mathbf{k}) \sim 1/(\mathbf{k}^2)^2$). However, an extrapolation in β will be necessary to determine the strength of this singularity in the continuum limit, because U_4 is quite far from the continuum limit at $\beta = 2.2$, as compared to U_i, for $i = 1, 2, 3$, due to the gauge-fixing in the minimal Coulomb gauge that is described in footnote 1. In the present Letter we present a new fit to $D^{\mathrm{tr}}(\mathbf{k}, L)$, and most importantly, its extrapolation to infinite lattice volume $L \to \infty$, using a lattice discretization of Gribov's formula.

2. Fit to $D^{\mathrm{tr}}(\mathbf{k})$

To parametrize the data we choose a fitting formula which: (i) gives a good fit to $D^{\mathrm{tr}}(\mathbf{k}, L)$ for all momenta $\mathbf{k}$ and volumes L^4, (ii) includes Gribov's formula (1) as a special case, and (iii) is physically transparent. Recall that a free continuum field with mass m has the equal-time propagator $(2\pi)^{-1} \int dk_4\, (k_4^2 + \mathbf{k}^2 + m^2)^{-1} = [2(\mathbf{k}^2 + m^2)^{1/2}]^{-1}$. With this in mind, consider the continuum formula [3]

$$D^{\mathrm{tr}}(\mathbf{k}) = 4^{-1}\big[z(\mathbf{k}^2)^{\alpha} + s\big] \times \Big[(\mathbf{k}^2 + m_1^2)(\mathbf{k}^2 + m_2^2)\Big]^{-1/2} \tag{2}$$

with singularities corresponding to poles at m_1^2 and m_2^2, which we take to be either a pair of real numbers or a complex-conjugate pair. [4] For complex-conjugate poles $m_1^2 = m_2^{*2} = x + iy$, this reads

$$D^{\mathrm{tr}}(\mathbf{k}) = 4^{-1}\big[z(\mathbf{k}^2)^{\alpha} + s\big]\big[(\mathbf{k}^2 + x)^2 + y^2\big]^{-1/2}. \tag{3}$$

The case of a pair of real poles is obtained from this formula by taking $y^2 < 0$. Gribov's formula (1) is recovered for $\alpha = 0.5$, $s = 0$, $x = 0$, and $y = M^2$ (and $z = 2$), corresponding to a pair of purely imaginary poles.

To obtain a lattice discretization of (2), note that for $k_\mu = 2\sin(\theta_\mu/2)$, the lattice free propagator at equal time is given by $(2\pi)^{-1} \int_0^{2\pi} d\theta_4\, (k_4^2 + \mathbf{k}^2 + m_1^2)^{-1} = (4h_1 + h_1^2)^{-1/2}$, where $h_1 \equiv \mathbf{k}^2 + m_1^2$. This suggests discretizing (2) by the substitution, $4h_1 \to 4h_1 + h_1^2$, and similarly for $1 \to 2$. For $m_1^2 = m_2^{*2} = x + iy$, one has $4h_1 + h_1^2 = u + 2iyv$, where $u \equiv 4(\mathbf{k}^2 + x) + (\mathbf{k}^2 + x)^2 - y^2$, and $v \equiv (2 + \mathbf{k}^2 + x)$. This gives the lattice fitting formula that we shall use,

$$D^{\mathrm{tr}}(\mathbf{k}) = \big[z(\mathbf{k}^2)^{\alpha} + s\big]\big(u^2 + 4y^2v^2\big)^{-1/2}, \tag{4}$$

where $k_i = 2\sin(\theta_i/2)$, and $-\pi \leqslant \theta_i = 2\pi n_i/L \leqslant \pi$, for integer n_i. In the continuum limit one has $u \to 4(\mathbf{k}^2 + x)$ and $v \to 2$, and (4) approaches (3).

For each lattice size L we have made a fit of the parameters $z(L), s(L), \alpha(L), x(L)$ and $y^2(L)$ for the nine lattice volumes considered. Fig. 2 shows our fit for $D^{\mathrm{tr}}(\mathbf{k})$ for $L = 18, 24, 30$, and the fit to the other volumes are comparable. There is no a priori reason why a 2-pole fit should be accurate over the whole range of momenta considered. [5] However, the fit is excellent for all momenta $\mathbf{k}^2$ and for each L. A striking feature of the fit is that $y^2(L)$ is positive for all 9 values of L, corresponding to a pair of *complex*-conjugate poles rather than a pair of real poles. Complex poles are also seen in the minimal Landau gauge [14] and for gauges that interpolate between them [15], and at finite temperature [15,16]. By contrast, in the maximal Abelian gauge [15–17] and in Laplacian gauge [18], poles are observed to occur at real k^2.

In order to do the crucial extrapolation to infinite lattice volume, we fit 5 curves [6] $z(L), s(L), \alpha(L), x(L)$

[3] In Ref. [12] we made a fit to a formula corresponding to a sum of 2 poles. However, it did not include the Gribov formula as a special case.

[4] According to the general principles of quantum field theory, the propagator of physical particles should have poles only at real positive m^2. However, in the confined phase the gluon propagator may have singularities that correspond to unphysical excitations.

[5] In our simulations we consider only 3-momenta aligned along major axes $\theta_i = (0, 0, 2\pi n/L)$. Thus, the maximum momentum value (for each lattice side L) is 2 in lattice units, and 1.876 GeV in physical units (with the inverse lattice spacing a^{-1} set equal to 0.938 GeV, see Ref. [12]).

[6] To facilitate comparison with $x(L)$, we plot $y(L) > 0$.

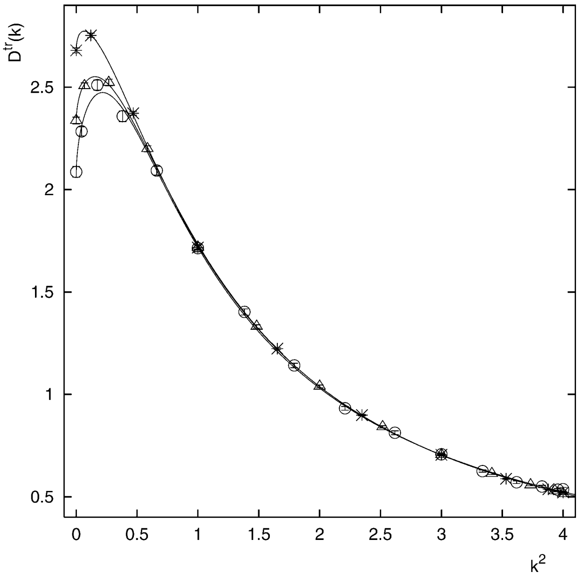

Fig. 2. Plot of the gluon propagator $D^{\mathrm{tr}}(\mathbf{k})$ as a function of the square of the lattice momentum $\mathbf{k}^2$ for $L = 18(*)$, $24(\triangle)$ and $30(\bigcirc)$ and fits of these data using Eq. (4). Error bars are one standard deviation.

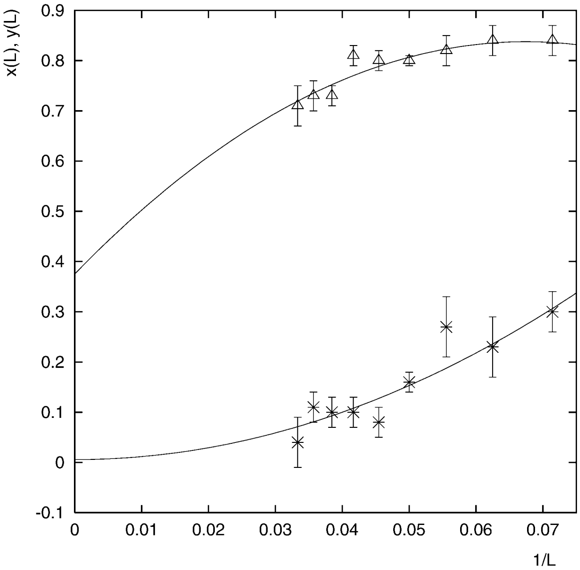

Fig. 4. Fits of the parameters $x(L)(*)$ and $y(L)(\triangle)$ plotted against $1/L$. For $x(L) = a + c/L^2$ we obtain $a = 0.006 \pm 0.024$, $c = 59.0 \pm 9.9$, and $\chi^2/\mathrm{d.o.f.} = 0.858$. For $y(L) = a + b/L + c/L^2$ we obtain $a = 0.375 \pm 0.162$, $b = 13.7 \pm 6.3$, $c = -101.6 \pm 61.0$, and $\chi^2/\mathrm{d.o.f.} = 1.024$.

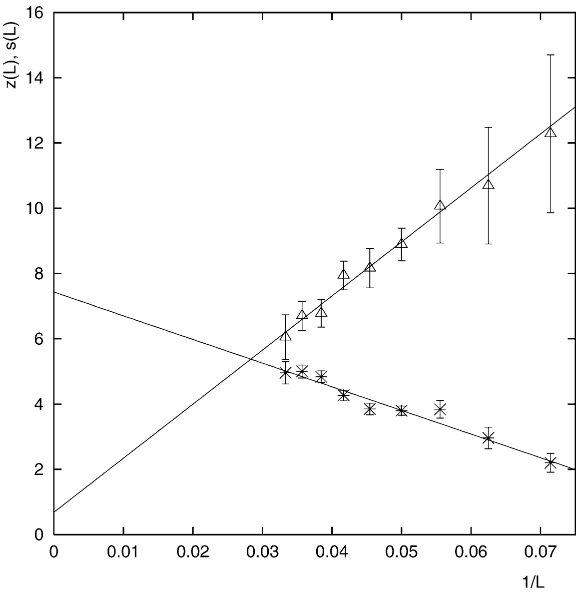

Fig. 3. Fits of the parameters $z(L)(*)$ and $s(L)(\triangle)$ plotted against $1/L$. For $z(L) = a + b/L$, we obtain $a = 7.43 \pm 0.36$, $b = -72.5 \pm 7.7$, and $\chi^2/\mathrm{d.o.f.} = 1.13$. For $s(L) = a + b/L$, we obtain $a = 0.685 \pm 0.534$, $b = 165.6 \pm 12.7$, and $\chi^2/\mathrm{d.o.f.} = 0.183$.

and $y(L)$ to our values of these fitting parameters. After some experimentation, we used the 3 fitting functions $a+b/L$, $a+b/L+c/L^2$, and $a+c/L^2$, for each of the 5 parameters. Of these 3 fits, we have displayed, for each parameter, the one with the smallest χ^2 per degree of freedom, with values reported in the captions of Figs. 3–5.

3. Conclusions

(1) One sees clearly from Fig. 2 that the equal-time would-be physical gluon propagator $D^{\mathrm{tr}}(\mathbf{k}, L)$ passes through a maximum and *decreases* as the momentum $\mathbf{k}^2$ approaches $\mathbf{0}$ (for fixed L). The decrease is more pronounced as the size L of the lattice increases. This counter-intuitive behavior is direct evidence of the suppression of low-momentum components of configurations $A^{\mathrm{tr}}(\mathbf{k})$ caused by the restriction to the Gribov region.

(2) In Fig. 3, the extrapolated value of $s(\infty) = 0.685 \pm 0.534$ is sufficiently close to 0 (compared to $b/L \sim 6$ to 12) that we feel justified in concluding that

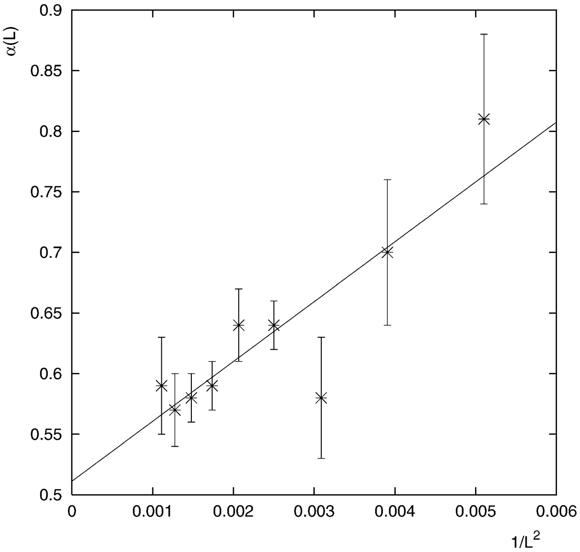

Fig. 5. Fit of the parameter $\alpha(L)$ plotted against $1/L^2$. For $\alpha = a + c/L^2$, we obtain $a = 0.511 \pm 0.022$, $c = 49.4 \pm 10.5$, and $\chi^2/\text{d.o.f.} = 0.666$.

the gluon propagator $D^{\text{tr}}(\mathbf{k})$ extrapolates to a value compatible with *zero* at $\mathbf{k} = \mathbf{0}$.

(3) As a result, at infinite volume our fitting formula behaves like $D^{\text{tr}}(\mathbf{k}) \propto (\mathbf{k}^2)^{\alpha}$, at low momentum. The extrapolated value of $\alpha(L)$ at infinite volume, $\alpha = 0.511 \pm 0.022$, is in striking numerical agreement with Gribov's formula (1), which gives $\alpha = 0.5$.

(4) We have obtained an excellent 2-pole fit for $D^{\text{tr}}(\mathbf{k}, L)$. Our fit indicates that the poles occur at *complex* $m^2 = x(L) \pm iy(L)$. The real part $x(L)$ extrapolates to 0.006 ± 0.024 at $L = \infty$. Remarkably, a pair of poles in $\mathbf{k}^2$ at purely imaginary $m^2 = 0 \pm iy$ agrees with the Gribov equal-time propagator $D^{\text{tr}}(\mathbf{k}) = (2|\mathbf{k}|)^{-1}[1 + M^4(\mathbf{k}^2)^{-2}]^{-1/2}$, with $y^2 = M^4$. Moreover at large $\mathbf{k}$, this gives a leading correction to the free equal-time propagator of relative order $(\mathbf{k}^2)^{-2}$ with coefficient of dimension $(\text{mass})^4$. It may not be a coincidence that this is the dimension of the gluon condensate $\langle F^2 \rangle$, which is the lowest dimensional condensate in QCD. Because of the theoretical suggestiveness of our result, we are encouraged to report the values $m^2 = 0 \pm iy$, for $y = 0.375 \pm 0.162$ in lattice units, or $y = 0.330 \pm 0.142\ \text{GeV}^2$, $M = y^{1/2} = 0.575 \pm 0.124$ GeV for the location of the gluon poles in $\mathbf{k}^2$. Here it is assumed that we are already in the scaling region for $D^{\text{tr}}(\mathbf{k})$, which remains to be verified by further numerical studies. That this is not entirely unreasonable is suggested by the fact that scaling for the Landau gauge propagator in $SU(2)$ has been observed [8] in the range $2.1 \leqslant \beta \leqslant 2.6$. Note also that Gribov derived his formula in the continuum theory, namely, at infinite β, so on the theoretical side it is to be expected that our fit will remain valid throughout the scaling region.

(5) The observed strong *enhancement* [12] of the instantaneous color-Coulomb potential $\widetilde{V}(\mathbf{k})$ and the observed *suppression* of the equal-time would-be physical gluon propagator $D^{\text{tr}}(\mathbf{k})$ both at low $\mathbf{k}$, strongly support the confinement scenario of Gribov [1,2]. In addition to this qualitative agreement, we note excellent numerical agreement of our fit to the formula $D^{\text{tr}}(\mathbf{k}) = z(|\mathbf{k}|/2)[(\mathbf{k}^2)^2 + M^4]^{-1/2}$. Although this formula cannot be exact, it appears that deviations from it are relatively weak at both high- and low-momentum regimes. If this excellent fit is maintained at larger β values, then it appears that we have obtained a quantitative understanding of $D^{\text{tr}}(\mathbf{k})$.

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